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Solvable structures and hidden symmetries

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Abstract. We show that some examples in the literature of non-standard symmetry reductions of ordinary differential equations can be understood using the concept of a solvable structure of one-forms.

1. Introduction

The presence of a solvable Lie algebra of Lie symmetries of a set of differential equations allows the systematic reduction of the order of the system and thus its integrability by quadrature. That such a condition is sufficient, but not necessary, for integrability by quadratures is well known and variants of the 'classical' approach are available, enabling one to search for exceptions.

In this paper, we wish to focus on one particular class of examples of a kind Olver illustrated, by way of warning, in [1]. The case at hand occurs when the system obtained by symmetry reduction possesses a symmetry which is not a symmetry of the unreduced system. In [2], Guo and Abraham-Schrauner enact a systematic search for examples of this kind by starting with a separable first-order ordinary differential equation (ODE) and *extending it to a second-order* ODE *using the differential invariants of a Lie symmetry*. At the end of the paper, they give a table of second-order ODEs, obtained by this method, the members of which possess only one Lie symmetry but acquire a new symmetry upon reduction and are therefore solvable by quadratures. The symmetries of the reduced ODEs are referred to as type II *hidden symmetries* of the unreduced ODEs.

Our purpose here is to show that Olver's example, and those in [2], are all simply understood by recourse to a generalization of solvable algebras called *solvable structures*. This name is due to Basarab-Horwath who has given a general description of their integrable properties in [3].

In this paper, we describe solvable structures and their relationship with ODEs. After stating a precise definition of solvable structures in terms of vector fields which is equivalent to the one given by Basarab-Horwath in [3], we develop the dual picture in terms of one-forms and re-establish some of Basarab-Horwath's results from that point of view. In the next section, we spell out the relationship between solvable structures and ODEs. In the concluding section, we exhibit an example calculation for the equation mentioned by Olver in [1] and give the results for all the cases given by Guo and Abraham-Schrauner in [2].

Throughout the paper we assume that we are working on an open simply connected subset D^n of $\mathbb{R}^n$. Functions are usually assumed to be smooth and well defined on D^n. The phrase '(linearly) independent on D^n' means pointwise linearly independent at every point of D^n. This notion of linear independence implies that independent vector fields or

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one-forms are not allowed to vanish at any point of D^n. Where necessary, the reader should assume that the domains have been restricted accordingly.

2. Solvable structures

Before we start, we have to introduce some notation and definitions of modern differential geometry (such as in [5]). Let $i_X\omega$ denote the interior product (contraction) of a differential form ω (completely anti-symmetric contravariant tensor) with a vector field X. Further, let $\mathrm{d}\omega$ denote the exterior derivative of a differential form. A function is interpreted as a zero-form, its exterior derivative being given by the differential of the function. In addition to that, we will only require the exterior derivative of a one-form, which can be defined in the following way:

$$\mathrm{d}\omega\,(X,Y) = X(i_Y\omega) - Y(i_X\omega) - i_{[X,Y]}\omega \tag{1}$$

for any one-form ω and arbitrary vector fields X and Y. Also, let $.\wedge.$ denote the exterior product of two forms or two vectors (completely anti-symmetrized tensor product). The following identities hold:

$$\begin{aligned}
&i_X^2 = 0\\
&i_X(\omega_1\wedge\omega_2) = i_X\omega_1\wedge\omega_2 + (-1)^p\omega_1\wedge i_X\omega_2\\
&i_{X\wedge Y}\omega = i_X i_Y\omega\\
&i_X f = 0\\
&i_X\,\mathrm{d}f = Xf
\end{aligned}$$

for all functions f, differential forms ω,ω_2 and p-forms ω_1 and all vector fields X, Y. The Lie derivative $\mathcal{L}_X$ of a form with respect to the vector field X can be defined as

$$\mathcal{L}_X = \mathrm{d}i_X + i_X\mathrm{d} \tag{2}$$

and the following identities hold:

$$\begin{aligned}
&\mathcal{L}_X(f\omega) = f\mathcal{L}_X\omega + (Xf)\omega\\
&\mathcal{L}_X(\omega_1\wedge\omega_2) = (\mathcal{L}_X\omega_1)\wedge\omega_2 + \omega_1\wedge(\mathcal{L}_X\omega_2)
\end{aligned}$$

for all functions f, differential forms ω,ω_1,ω_2 and vector fields X. The Lie derivative $\mathcal{L}_XY$ of a vector field Y with respect to the vector field X is the same as the commutator $[X,Y]$. The following commutation rule for $\mathcal{L}_X$ and i_Y holds when they are acting on a differential form

$$\mathcal{L}_X i_Y - i_Y\mathcal{L}_X = i_{[X,Y]}. \tag{3}$$

Now, let $\boldsymbol{X} = \{X_1, X_2, \ldots, X_r\}$ be a system of independent vector fields on D^n which are in involution, i.e. there exist smooth functions c_{ij}^k, $i,j,k = 1,\ldots,r$ such that $[X_i, X_j] = c_{ij}^k X_k$ where we employ the summation convention for repeated indices.

Definition 1. A nowhere-vanishing vector field Y on D^n is called a symmetry of an involutive system of independent vector fields $\boldsymbol{X}$, as just described, iff the following conditions hold:

(i) $X_1, \ldots, X_r, Y$ are independent; and

(ii) there exist smooth functions $c_i^k (k, i = 1, \ldots, r)$ such that $[X_i, Y] = c_i^k X_k$.

We would like to point out that this definition represents a generalization of the idea of a Lie point symmetry. Here, and in the following, the term system is used loosely and simply refers to a collection of objects (vector fields or one-forms). Given a system $\boldsymbol{S}$, span $\boldsymbol{S}$ stands for the $\mathfrak{F}(D^n)$-linear space obtained by taking $\mathfrak{F}(D^n)$-linear combinations of the objects in $\boldsymbol{S}$ where $\mathfrak{F}(D^n)$ is the ring of all smooth functions on D^n. Two systems, $\boldsymbol{S}$ and $\widetilde{\boldsymbol{S}}$, are called equivalent if the objects in $\widetilde{\boldsymbol{S}}$ can be expressed as non-singular (invertible) $\mathfrak{F}(D^n)$-linear combinations of the objects in $\boldsymbol{S}$ at every point of D^n. Note that span $\boldsymbol{S} =$ span $\widetilde{\boldsymbol{S}}$.

Let $\boldsymbol{\Omega} = \{\omega^1, \ldots, \omega^s\}$ be a system of independent one-forms on D^n which is closed, i.e. there exist smooth one-forms $\lambda_l^k (k, l = 1, \ldots, s)$ such that $\mathrm{d}\omega^k = \lambda_l^k \wedge \omega^l$.

Definition 2. A nowhere-vanishing vector field Y on D^n is called a symmetry of a closed system of independent one-forms $\boldsymbol{\Omega}$, as just described, iff the following conditions hold:

(i) at every point in D^n there exists $j (1 \leqslant j \leqslant s)$ such that $i_Y \omega^j \neq 0$; and

(ii) there exist smooth functions $c_l^k (k, l = 1, \ldots, s)$ such that $\mathcal{L}_Y \omega^k = c_l^k \omega^l$.

For every system of independent vector fields $\boldsymbol{X} = \{X_1, \ldots, X_r\}$ there exists a corresponding system of independent one-forms $\boldsymbol{\Omega} = \{\omega^1, \ldots, \omega^{n-r}\}$ which is necessarily and sufficiently characterized by

$$i_{X_k} \omega^l = 0 \qquad (k = 1, \ldots, r; l = 1, \ldots, n - r). \tag{4}$$

Equation (4) is obviously not affected by replacing $\boldsymbol{X}$ or $\boldsymbol{\Omega}$ with equivalent systems $\tilde{\boldsymbol{X}}$ or $\tilde{\boldsymbol{\Omega}}$, respectively. The linearity properties of the interior product also imply that

$$i_{X_k} \omega = 0 (\forall k = 1, \ldots, r) \Leftrightarrow \omega \in \operatorname{span} \boldsymbol{\Omega} \tag{5}$$

$$i_Y \omega^l = 0 (\forall l = 1, \ldots, n - r) \Leftrightarrow Y \in \operatorname{span} \boldsymbol{X} \tag{6}$$

for all one-forms ω and vector fields Y. Now, let $X, Y \in \operatorname{span} \boldsymbol{X}$ and $\omega \in \operatorname{span} \boldsymbol{\Omega}$. Consider $i_{[X,Y]}\omega$; using the commutation rule (3) and equations (2), (5) and (6) we obtain

$$i_{[X,Y]}\omega = \mathrm{d}\omega\,(X, Y). \tag{7}$$

From this we can deduce the following lemma.

Lemma 1. Let $\boldsymbol{X}$ and $\boldsymbol{\Omega}$ be corresponding systems of independent vector fields and one-forms, respectively, as just described. Then, $\boldsymbol{X}$ is involutive iff $\boldsymbol{\Omega}$ is closed.

We also have the following result.

Lemma 2. Let $\boldsymbol{X}$ be an involutive system of independent vector fields and let Ω be a corresponding closed system of one-forms. A vector field Y is a symmetry of $\boldsymbol{X}$ iff Y is a symmetry of $\boldsymbol{\Omega}$.

Proof. Let $\boldsymbol{X} = \{X_1, \ldots, X_r\}$ and $\boldsymbol{\Omega} = \{\omega^1, \ldots, \omega^{n-r}\}$. Note that by equation (4) $i_{X_k}\omega^l = 0 (k = 1, \ldots, r; l = 1, \ldots, n-r)$. Hence, using commutation relation (3), we obtain

$$i_{[Y, X_k]}\omega^l = -i_{X_k}\mathcal{L}_Y\omega^l. \tag{8}$$

If Y is a symmetry of $\boldsymbol{X}$ then the left-hand side equals 0 by (4) and using (6) we deduce that Y is a symmetry of $\boldsymbol{\Omega}$. If Y is a symmetry of $\boldsymbol{\Omega}$ then the right-hand side equals 0 by (4) and using (6) we deduce that Y is a symmetry of $\boldsymbol{X}$, which completes the proof. We would like to point out that part (i) of definition 1 directly corresponds to part (i) of definition 2. □

Let $\{\boldsymbol{X}, \boldsymbol{Y}_l\} = \{X_1, \ldots, X_r, Y_1, \ldots, Y_l\}$ and let $\{\boldsymbol{X}, \boldsymbol{Y}_0\} = \boldsymbol{X}$. We can now define what a solvable structure is.

Definition 3. We say that the system $\{\boldsymbol{X}, \boldsymbol{Y}_{n-r}\}$ is a solvable structure with respect to the involutive system $\boldsymbol{X}$ iff $Y_l (l = 1, \ldots, n-r)$ is a symmetry of the system $\{\boldsymbol{X}, \boldsymbol{Y}_{l-1}\}$.

Note that, in general, the Y_l do *not* form a solvable algebra.

Let us now construct a dual version of the solvable stucture $\{\boldsymbol{X}, \boldsymbol{Y}_{n-r}\}$ in terms of one-forms. Let $\omega^1, \ldots, \omega^{n-r}$ be a set of independent one-forms such that $i_{Y_s}\omega^s = 1$ (no summation) and $\omega^s = 0$ on $\{\boldsymbol{X}, Y_1, \ldots, \widehat{Y_s}, \ldots, Y_{n-r}\}$ where $\widehat{Y_s}$ denotes omission of Y_s $(s = 1, \ldots, n-r)$. Before proceeding let us introduce the following notation. Let $\mathcal{I}(\{\omega^1, \ldots, \omega^s\})$ be the ideal generated by $\omega^1, \ldots, \omega^s$ undertaking exterior products and let $\mathcal{I}(\emptyset) = \{0\}$. The system $\{\omega^1, \ldots, \omega^{n-r}\}$ has distinguishing closure properties undertaking exterior derivatives:

Proposition 3. For all $s = 1, \ldots, n-r$, we have

$$\mathrm{d}\omega^s \in \mathcal{I}(\{\omega^{s+1}, \ldots, \omega^{n-r}\}).$$

Proof. From the way in which the ω^s are defined, it immediately follows that $\{\boldsymbol{X}, \boldsymbol{Y}_s\}$ and $\{\omega^{s+1}, \ldots, \omega^{n-r}\}$ are corresponding systems of vector fields and one-forms, respectively. Hence, in order to prove proposition 3, it is sufficient to show that $\mathrm{d}\omega^s(V, W) = 0$ for arbitrary elements V and W of $\mathrm{span}\{\boldsymbol{X}, \boldsymbol{Y}_s\}$. Using equation (1) we obtain

$$\mathrm{d}\omega^s(V, W) = V(i_W\omega^s) - V(i_W\omega^s) - i_{[V,W]}\omega^s.$$

We continue by decomposing V and W with the help of ω^s. Throughout this proof, summation over repeated indices is *not* intended. Let $V_s = (i_V\omega^s)Y_s$ and $\overline{V}_s = V - V_s$. Similarly define W_s and $\overline{W}_s$. Clearly $\overline{V}_s$ and $\overline{W}_s$ are in $\mathrm{span}\{\boldsymbol{X}, \boldsymbol{Y}_{s-1}\}$ while V_s and W_s are multiples of Y_s. We also have $V = V_s + \overline{V}_s$ and $W = W_s + \overline{W}_s$. Putting these in the above equation, and taking account of the fact that ω^s kills everthing in $\mathrm{span}\{\boldsymbol{X}, \boldsymbol{Y}_{s-1}\}$, we are left with

$$\mathrm{d}\omega^s(V, W) = V(i_{W_s}\omega^s) - W(i_{V_s}\omega^s) - i_{[V_s,W_s]+[V_s,\overline{W}_s]+[\overline{V}_s,W_s]}\omega^s. \tag{9}$$

Observe that $[V_s, W_s] = V_s(i_{W_s}\omega^s)Y_s - W_s(i_{V_s}\omega^s)Y_s$, $[V_s, \overline{W}_s] = (i_{V_s}\omega^s)[Y_s, \overline{W}_s] - \overline{W}_s(i_{V_s}\omega^s)Y_s$ and $[\overline{V}_s, W_s] = \overline{V}_s(i_{W_s}\omega^s)Y_s + (i_{W_s}\omega^s)[\overline{V}_s, Y_s]$. Furthermore, note that

$[Y_s, \overline{W}_s]$ and $[\overline{V}_s, Y_s]$ are elements of span$\{\boldsymbol{X}, \boldsymbol{Y}_{s-1}\}$ since Y_s is a symmetry of the system $\{\boldsymbol{X}, \boldsymbol{Y}_{s-1}\}$. Evaluating the inner product in equation (9), therefore, results in

$$\begin{aligned} i_{[V_s,W_s]+[V_s,\overline{W}_s]+[\overline{V}_s,W_s]}\omega^s &= V_s(i_{W_s}\omega^s) - W_s(i_{V_s}\omega^s) - \overline{W}_s(i_{V_s}\omega^s) + \overline{V}_s(i_{W_s}\omega^s) \\ &= V(i_{W_s}\omega^s) - W(i_{V_s}\omega^s). \end{aligned}$$

Putting this in equation (9) finally gives us the required result

$$\mathrm{d}\omega^s\,(V, W) = 0$$

which completes the proof. □

There is a converse to the last proposition.

Proposition 4. A system of independent one-forms $\omega^1, \ldots, \omega^{n-r}$ with the property that $\mathrm{d}\omega^s \in \mathcal{I}(\{\omega^{s+1}, \ldots, \omega^{n-r}\})$ for all $s = 1, \ldots, n-r$ corresponds to a solvable structure $\{\boldsymbol{X}, \boldsymbol{Y}_{n-r}\}$.

Proof. Let $\boldsymbol{X} = \{X_1, \ldots, X_r\}$ be a system of independent vector fields which corresponds to the closed system of independent one-forms formed by $\omega^1, \ldots, \omega^{n-r}$. Define Y_s $(s = 1, \ldots, n-r)$ by requiring that thoughout D^n, $i_{Y_j}\omega^s = 1$ if $j = s$ and $i_{Y_j}\omega^s = 0$ for $j \neq s$ $(j = 1, \ldots, n-r)$. By lemma 1 the system $\boldsymbol{X}$ is clearly involutive since it corresponds by definition to the closed system formed by the ω^s. Furthermore, the definition of the Y_s guarantees that they are independent of each other and the system $\boldsymbol{X}$. Again, throughout this proof, summation over repeated indices is *not* intended. In order to prove that Y_s is a symmetry of $\{\boldsymbol{X}, \boldsymbol{Y}_{s-1}\}$, we make use of lemma 2 and show that Y_s is a symmetry of $\{\omega^s, \ldots, \omega^{n-r}\}$. For the latter, by equation (5), it is sufficient to show that $i_X\mathcal{L}_{Y_s}\omega^j = 0$ for all $j = s, \ldots, n-r$ and any $X \in$ span$\{\boldsymbol{X}, \boldsymbol{Y}_{s-1}\}$. Using equation (2) we get

$$i_X\mathcal{L}_{Y_s}\omega^j = i_X\,\mathrm{d}(i_{Y_s}\omega^j) + i_X(i_{Y_s}\,\mathrm{d}\omega^j).$$

But $i_{Y_s}\omega^j$ is 1 or 0 and $\mathrm{d}\omega^j$ is certainly in $\mathcal{I}(\{\omega^{s+1}, \ldots, \omega^{n-r}\})$ because of the closure properies of the ω^s and, therefore, vanishes if it is applied to two vectors from span$\{\boldsymbol{X}, \boldsymbol{Y}_s\}$. Hence, we can deduce

$$i_X\mathcal{L}_{Y_s}\omega^j = 0$$

which completes the proof. □

Given the correspondence established by propositions 3 and 4, it makes sense to say that a system of one-forms which have the closure properties assumed in proposition 4 is a solvable structure in its own right. For this reason we are led to introduce the following definition.

Definition 4. The system of independent one-forms $\omega^1, \ldots, \omega^k$ $(1 \leqslant k \leqslant n)$ on D^n form a solvable structure iff for all $j = 1, \ldots, k$

$$\mathrm{d}\omega^j \in \mathcal{I}(\{\omega^{(j+1)}, \ldots, \omega^k\}).$$

By integrating a closed system $\boldsymbol{\Omega}$ of k independent one-forms we mean finding a system of coordinates for D^n such that keeping k of the coordinates constant gives a submanifold of D^n on which $\boldsymbol{\Omega}$ vanishes. By integrating an involutive system $\boldsymbol{X}$ of r independent vector fields we mean finding a coordinate system for D^n such that $\boldsymbol{X}$ is tangential to a submanifold of D^n defined by keeping $n-r$ of the coordinates constant. In both cases, such coordinate systems can generally only be defined locally in the neighbourhood of regular points. A moment of reflection reveals that integrating a closed system of one-forms is equivalent to integrating a corresponding involutive system of vector fields and *vice versa*. The same applies to equivalent systems.

Proposition 5. Solvable structures of one-forms or vector fields can be integrated, at least locally, by quadratures alone.

Proof. The vector-field case can be reduced to the one-form case by integrating a corresponding solvable structure of one-forms such as the one we constructed in the proof of proposition 3. Now, let $\omega^1, \ldots, \omega^k$ form a solvable structure of one-forms. In particular, this implies that $\mathrm{d}\omega^k$ equals zero. Therefore ω^k can be integrated by quadrature since D^n is simply connected. If one restricts attention to the submanifold defined by keeping $\int \omega^k$ constant, one finds that the restrictions of $\omega^1, \ldots, \omega^{k-1}$ form a solvable structure on this submanifold. One can, therefore, integrate the restriction of $\mathrm{d}\omega^{k-1}$ on this submanifold by quadrature. We can continue in this fashion by further restricting the submanifolds at each stage until we have finally integrated the $(k-1)$th restriction of ω^1. At this point we have fully integrated the system of one-forms. □

We would like to point out that a solvable structure, as introduced here, is a purely geometrical concept concerned with the integrability by quadratures of certain systems of vector fields or one-forms. Solvable structures may prove useful in the study of differential equations wherever the problem at hand can be reformulated as the task of integrating an integrable system of vector fields or one-forms. We will show how this is done for ODEs in the next section. The generalization of our approach to systems of ODEs is straightforward. The geometrical representation of partial-differential equations (PDE)s with more than one independent variable generally does not lead to an integrable system of one-forms because one has to append higher-order forms to achieve closure on taking exterior derivatives. Nevertheless, there is scope for utilizing solvable structures in the process of solving PDEs for tasks such as the integration of characteristic equations or determining the invariants of flows.

One might suspect that solvable structures should somehow fit into the framework of non-classical or conditional symmetries (the concept of 'non-classical' symmetries was introduced by W Bluman and J D Cole in 1969; since then, a vast number of publications on the subject has been produced and we refer the interested reader to a selection of recently published books for further reading and references; cf [6]). There are variations on the theme but basically a conditional symmetry does not preserve the original system of PDEs but an augmented system which consists of the original system together with some additional equations. The symmetries we encounter in the context of solvable structures generally do not preserve the original system either, but this time the system preserved by the symmetry is obtained by weakening the constraints of the original system and *not* by adding new ones. It, therefore, becomes clear that the two concepts are quite different.

3. Solvable structures and ODEs

Before we can apply the theory developed above to ODEs, we have to develop an appropriate geometric representation of a general nth-order ODE. Let the ODE be given by

$$F(x, y, y^{(1)}, \ldots, y^{(n)}) = 0 \tag{10}$$

for some smooth function F on a open subset D^{n+2} of $\mathbb{R}^{n+2}$, where $y^{(j)} = \mathrm{d}^j y/\mathrm{d}x^j$ $(j = 1, \ldots, n)$. Any solution $y(x)$ corresponds to a one-dimensional submanifold of D^{n+2}, where $x, y, y^{(j)}$ are now treated as independent coordinates on D^{n+2}. Conversely, making use of Cartan's work, we know that a one-dimensional submanifold Σ of D^{n+2}, which contains a generic point of the submanifold defined by $F = 0$, corresponds to a solution of equation (10) if the system of one-forms Ω_F formed by dF and the contact forms $\alpha^j = dy^{(j)} - y^{(j+1)}\,dx$ $(j = 0, \ldots, (n-1),\ y^{(0)} = y)$ vanish on Σ. Provided dF is independent of the contact forms (this is always true if $\partial F/\partial y^{(n)} \neq 0$), Ω_F is automatically closed because any system of $n+1$ independent one-forms on D^{n+2} is closed. Hence, we do not have to worry about integrability conditions. Since Ω_F is closed, the existence of solutions near a generic point $\underline{x}$ such that $F(\underline{x}) = 0$ is guaranteed by Frobenius' theorem. Let $\Omega = \{\omega^1, \ldots, \omega^{n+1}\}$ be a solvable structure of one-forms. If Ω is equivalent to Ω_F then integrating Ω is equivalent to integrating Ω_F. Now, consider the resulting coordinate system in which integral manifolds to Ω_F are obtained by keeping $n+1$ coordinates constant. When expressed as functions of $x, y, y^{(1)}, \ldots, y^{(n)}$, the constant coordinates can be interpreted as $n+1$ independent first integrals of the ODE given by equation (10) and, by inverting the whole system, one obtains y as a function of x and the constants, which gives a general solution of equation (10). For this reason, applying proposition 5 gives us the following preposition.

Proposition 6. Given a solvable structure Ω such that Ω is equivalent to Ω_F, as just described, we can locally solve the ODE given by equation (10) by quadratures alone.

We would like to point out that this method of using generalized symmetries for the complete integration of ODEs is not found in the standard literature on the subject. It is, however, indicated by Basarab-Horwath in [3].

Let us now turn to the question of how to find a solvable structure which is equivalent to a given closed system of one-forms. In theory, every closed system Ω is equivalent to some solvable structure. To see this, consider any coordinate system in which the integral submanifolds of Ω are obtained by keeping an appropriate number of these coordinate functions constant. The differentials of these coordinate functions are obviously all exact and therefore can be thought of as forming a solvable structure. Given that degree of generality, one should *not* expect that one will always be able to find a solvable structure equivalent to a given closed system of one-forms. Bearing this caveat in mind, the procedure for attempting to find a solvable structure for a given closed system of one-forms is straightforward. Let $\Omega_X = \{\alpha_1^1, \ldots, \alpha_1^{n-r}\}$ be a closed system of independent one-forms and let $\boldsymbol{X} = \{X_1, \ldots, X_r\}$ be a corresponding system of independent vector fields which are in involution. First, we try to find a symmetry Y_1 of Ω_X. Alternatively, by lemma 2, one can find Y_1 by requiring that it is a symmetry of $\boldsymbol{X}$. If successful, we can choose ω^1 and $\alpha_2^1, \ldots, \alpha_2^{n-r-1}$ which form a system of independent one-forms equivalent to Ω_X and have the following properties: $i_{Y_1}\omega^1 = 1$ and $\Omega_{\{X,Y_1\}} = \{\alpha_2^1, \ldots, \alpha_2^{n-r-1}\}$ corresponds to $\{\boldsymbol{X}, Y_1\}$. By adapting the proof of proposition 3, one can show that $d\omega^1 \in \mathcal{I}(\Omega_{\{X,Y_1\}})$. It is easy to see that by carrying on, if possible, one will eventually end up with a solvable structure of one-forms $\{\omega^1, \ldots, \omega^{n-r}\}$ which is equivalent to Ω_X and a corresponding solvable structure of vector fields $\{\boldsymbol{X}, Y_{n-r}\}$. If convenient, one can also first determine the solvable structure of vector fields $\{\boldsymbol{X}, Y_{n-r}\}$ and then obtain the ω^s by requiring that they are dual to the Y_s $(s = 1, \ldots, n-r)$.

We now want to explain the relationship between the solvable-structure approach and the classical Lie point symmetry reduction. Let Y_1 be a Lie point symmetry of the ODE given by equation (10). A Lie point symmetry corresponds to a point transformation in the space of the independent and dependent variables. In our case, this implies that there exist functions $\tilde{x} = \tilde{x}(x, y)$ and $\tilde{y} = \tilde{y}(x, y)$ such that $Y_1(\tilde{x}) = 0$ and $Y_1(\tilde{y}) = 1$. In the classical approach, the reduction is achieved by determining these functions $\tilde{x}$ and $\tilde{y}$ and then transforming to new coordinates $\tilde{x}, \tilde{y}, \tilde{y}^{(j)} = \mathrm{d}^j\tilde{y}/\mathrm{d}\tilde{x}^j$ $(j = 1, \ldots, n)$. Now consider the system $\widetilde{\Omega}_F$, formed by $\mathrm{d}F$ and the contact forms $\tilde{\alpha}^j = \mathrm{d}\tilde{y}^{(j)} - \tilde{y}^{(j+1)}\,\mathrm{d}\tilde{x}$ $(j = 0, \ldots, n-1,\ \tilde{y}^{(0)} = \tilde{y})$. Since Y_1 is a Lie point symmetry, the $\tilde{\alpha}^j$ can be expressed as nowhere-vanishing invertible $\mathfrak{F}(D^{n+2})$-linear combinations of the α^j (the original contact forms) and by definition $Y_1(F) = 0$ on $F = 0$. In the new coordinates, Y_1 is equal to the coordinate vector field corresponding to the $\tilde{y}$ coordinate and in the light of the last sentence we see that $\widetilde{\Omega}_F$ is equivalent to Ω_F, that $i_{Y_1}\tilde{\alpha}^j$ equals 1 if $j = 0$ and 0 otherwise and, finally, that $i_{Y_1}\,\mathrm{d}F = 0$ on $F = 0$. The final step in the classical reduction procedure is to observe that $\mathrm{d}F$ and $\tilde{\alpha}^1, \ldots, \tilde{\alpha}^{n-1}$ are completely independent of $\tilde{y}$ on $F = 0$ and, therefore, can be interpreted as a system of one-forms corresponding to an $(n-1)$th-order ODE with $\tilde{x}$ and $\tilde{y}^{(1)}$ as independent and dependent variables, respectively. From the solvable-structure point of view, one sees that Y_1 makes a perfect first vector field in a solvable structure and that $\tilde{\alpha}^0$ and the system formed by $\tilde{\alpha}^1, \ldots, \tilde{\alpha}^{n-1}$ and $\mathrm{d}F$ can be identified with ω^1 and the system $\Omega_{\{X, Y_1\}}$ as described above, respectively. While it is clear that Lie point symmetries of this and even further reduced systems can be used to form a solvable structure equivalent to the original system, there is no reason why those symmetries should be Lie point symmetries of the original system. This point will be illustrated in the following example section. Using the solvable structure approach it is not necessary to restrict one's attention to Lie point symmetries at any stage.

4. Examples

In this example section we want to demonstrate two points. First, that Olver's curious example is explained by the presence of a solvable structure and, taking this example, we will exemplify how the algorithm described above can be used to directly solve an ODE. Second, we will show how in the case of the examples constructed by Guo and Abraham-Schrauner, the (hidden) symmetry of the reduced ODE is systematically related to a solvable structure for the unreduced ODE.

Before we embark on the calculations, let us briefly mention that the geometric representation of an nth-order ODE, given by equation (10), can be simplified if F is given in solved form for the highest derivative; viz $F = y^{(n)} - f(x, y, \ldots, y^{(n-1)})$ for some smooth function f on $D^{(n+1)}$. In this case, one can replace the $y^{(n)}$ coordinate on $D^{(n+2)}$ with F and then restrict attention to the submanifold given by $F = 0$. In these coordinates, the contact forms α^j $(j = 0, \ldots, n-1)$ project rather conveniently into $F = 0$: the coordinate expressions for α^0 to α^{n-2} do not have to be altered at all and the expression for α^{n-1} becomes $\mathrm{d}y^{(n-1)} - f(x, y, \ldots, y^{(n-1)})\,\mathrm{d}x$. Actually these projected one-forms form a closed system of independent one-forms on D^{n+1} (given by $F = 0$) and one, therefore, has effectively reduced the order of the problem by one.

In solved form, Olver's example ([1], example 2.58, p 147) is given by the following equation

$$y^{(2)} = \frac{yy^{(1)} - x(y^{(1)})^2}{x^2}. \tag{11}$$

Restricting our attention to an appropriate domain, the associated system of one-forms $\{\alpha^0, \alpha^1\}$ on D^3 is given by

$$\alpha^0 = \mathrm{d}y - y^{(1)}\,\mathrm{d}x$$

$$\alpha^1 = \mathrm{d}y^{(1)} - \frac{y^{(1)} - x(y^{(1)})^2}{x^2}\,\mathrm{d}x.$$

We already know that this system has the vector field $Y_1 = x\partial_x + y\partial_y$ as symmetry and that Y_1 represents the only Lie point symmetry of the ODE given by equation (11). We now look for an equivalent system of one-forms $\{\omega^1, \alpha_1^1\}$ such that $i_{Y_1}\omega^1 = 1$ and $i_{Y_1}\alpha_1^1 = 0$. A possible choice is

$$\omega^1 = \frac{\mathrm{d}y - y^{(1)}\,\mathrm{d}x}{y - xy^{(1)}}$$

$$\alpha_1^1 = yy^{(1)}\mathrm{d}x - xy^{(1)}\,\mathrm{d}y - x^2\,\mathrm{d}y^{(1)}.$$

Our next task is to find a symmetry Y_2 of α_1^1. For the purpose of determining whether Y_2 is a symmetry, only the part of Y_2 that does not get killed by α_1^1 matters. Therefore, one may assume that $Y_2 = \gamma(x, y, y^{(1)})\partial_{y^{(1)}}$ for some smooth function γ. The determining equations are obtained by requiring that $\mathcal{L}_{Y_2}\alpha_1^1 = \lambda(x, y, y^{(1)})\alpha_1^1$. After some algebraic manipulations, one is left with the following system of linear first-order PDEs:

$$x\frac{\partial\gamma}{\partial x} + y\frac{\partial\gamma}{\partial y} = 0$$

$$\gamma + x\frac{\partial\gamma}{\partial y} - y^{(1)}\frac{\partial\gamma}{\partial y^{(1)}} = 0.$$

It is easy to see that $\gamma = y^{(1)}$ is a solution for this system. Contracting Y_2 with α_1^1 gives us the required integrating factor for α_1^1. We now have a solvable structure $\{\omega^1, \omega^2\}$ for the ODE, given by equation (11), where ω^2 is given by

$$\omega^2 = \frac{\alpha_1^1}{-x^2y^{(1)}} = \frac{yy^{(1)}\,\mathrm{d}x - xy^{(1)}\,\mathrm{d}y - x^2\,\mathrm{d}y^{(1)}}{-x^2y^{(1)}}.$$

In order to solve the ODE given by equation (11), we have to integrate the solvable structure $\{\omega^1, \omega^2\}$. Starting with ω^2, we obtain

$$\int\omega^2 = \frac{y}{x} + \ln|y^{(1)}|.$$

Solving for $y^{(1)}$ gives $y^{(1)} = \pm\exp(\int\omega^2)\exp(y/x)$. When restricting to the submanifold given by keeping $\int\omega^2$ constant, we can replace $\pm\exp(\int\omega^2)$ with some non-zero constant c. On this submanifold, ω^1 is given by

$$\omega^1 = \frac{\mathrm{d}y - c\mathrm{e}^{y/x}\,\mathrm{d}x}{y - cx\mathrm{e}^{-y/x}}$$

which can be integrated by substituting $v = y/x$ which leaves us with

$$\int\omega^1 = \int\frac{\mathrm{d}v}{v - c\mathrm{e}^{-v}} + \ln|x|.$$

The last line integral cannot be performed explicitly and, hence, one cannot explicitly solve it for y either. Nevertheless, keeping $\int\omega^1$ constant, the last equation can be regarded as a general implicit solution to the ODE given by equation (11) with the proviso that we obviously have not dealt with the singular solutions of the ODE.

Table 1. Solvable structures for the eight second-order ODEs constructed by Guo and Abraham–Schrauner in [2].

$h(x, y, p)$	ω^1	ω^2	$\mathrm{d}\omega^1$
$f(y)$	$-\frac{1}{p}\alpha^0$	$-f(y)\alpha^0 + p\alpha^1$	$-\frac{1}{p^3}\,\mathrm{d}y \wedge \omega^2$
$\frac{f(x)}{p}$	α^0	$p\alpha 1$	$\frac{1}{p}\,\mathrm{d}x \wedge \omega^2$
$\frac{f(y)}{x^2} - \frac{p}{x}$	$-\frac{1}{xp}\alpha^0$	$-f(y)\alpha^0 + x^2 p\alpha^1$	$-\frac{1}{x^3 p^3}\,\mathrm{d}y \wedge \omega^2$
$\frac{y^2}{p} f(x) + \frac{p^2}{y}$	$\frac{1}{y}\alpha^0$	$-\frac{p^2}{y^3}\alpha^0 + \frac{p}{y^2}\alpha^1$	$\frac{y}{p}\,\mathrm{d}x \wedge \omega^2$
$\frac{f(x)}{x(xp-y)}$	$\frac{1}{x}\alpha^0$	$(y-xp)\alpha^0 - x(y-xp)\alpha^1$	$\frac{1}{x^2(xp-y)}\,\mathrm{d}x \wedge \omega^2$
$\frac{p^4 f(y)}{y(xp-y)}$	$-\frac{1}{yp}\alpha^0$	$-\left\{\frac{xp-y}{p^2} + f(y)\right\}\alpha^0 + \frac{y(xp-y)}{p^3}\alpha^1$	$-\frac{p}{y^2(xp-y)}\,\mathrm{d}y \wedge \omega^2$
$-\frac{p^3}{y^3} f\left(\frac{x}{y}\right)$	$\frac{1}{y(y-xp)}\alpha^0$	$\left\{\frac{y-xp}{p^2} + \frac{x}{y^2} f\left(\frac{x}{y}\right)\right\}\alpha^0 - \frac{y(y-xp)}{p^3}\alpha^1$	$\frac{p^3}{y^2(y-xp)^3}(-y\,\mathrm{d}x + x\,\mathrm{d}y) \wedge \omega^2$
$\frac{1}{x^3} f\left(\frac{y}{x}\right)$	$\frac{1}{x(y-xp)}\alpha^0$	$\left\{y - xp - \frac{1}{x} f\left(\frac{y}{x}\right)\right\}\alpha^0 + x(y-xp)\alpha^1$	$\frac{1}{x^2(y-xp)^3}(-y\,\mathrm{d}x + x\,\mathrm{d}y) \wedge \omega^2$

In [2], Guo and Abraham-Schrauner systematically construct eight second-order ODEs which are generally only invariant under a one-parameter Lie point group but acquire a new symmetry when reduced to a first-order ODE (they also provide a list of special cases in which the second-order ODEs are invariant under a solvable two-parameter group). They start with the separable first-order ODE, which in solved form is given by

$$\frac{\mathrm{d}w}{\mathrm{d}v} = \frac{f(v)}{w} \tag{12}$$

where f is a smooth function of the independent variable v. This ODE is invariant under a point transformation represented by the vector field $(1/w)\partial_w$. The first-order ODE is then transformed to a second-order ODE using two invariants W and V of a Lie point symmetry represented by the vector field U_1. The transformation is given by

$$\begin{aligned} w &= W(x, y, y^{(1)}) \\ v &= V(x, y) \end{aligned} \tag{13}$$

where $U_1(W) = U_1(V) = 0$. Substituting this into equation (12) results in a second-order ODE which is invariant under U_1 but, in general, is not invariant under $(1/w)\partial_w$. Hence, $(1/w)\partial_w$ is called a hidden symmetry. One can, however, easily construct a solvable structure for the second-order ODE which is closely related to the transformation (13). Since (12) is given in closed form, it can be represented by

$$\alpha = \mathrm{d}w - \frac{f(v)}{w}\,\mathrm{d}v.$$

Note that α is usually not exact but contracting $(1/w)\partial_w$ with α gives the required integrating factor. Transforming α by (13) results in

$$\alpha = \left(\frac{\partial W}{\partial y} - \frac{f(V)}{W}\right)(\mathrm{d}y - y^{(1)}\,\mathrm{d}x) + \frac{\partial W}{\partial y^{(1)}}(\mathrm{d}y^{(1)} - h\,\mathrm{d}x)$$

where

$$h = -\left(\frac{\partial W}{\partial x} - \frac{f(V)}{W}\frac{\partial V}{\partial x} + y^{(1)}\left(\frac{\partial W}{\partial y} - \frac{f(V)}{W}\frac{\partial V}{\partial y}\right) \bigg/ \frac{\partial W}{\partial y^{(1)}}\right).$$

Observe that $\alpha^0 = \mathrm{d}y - y^{(1)}\,\mathrm{d}x$ and $\alpha^1 = \mathrm{d}y^{(1)} - h\,\mathrm{d}x$ represent the second-order ODE given in solved form by $y^{(2)} = h(x, y, y^{(1)})$. By construction, U_1 is a symmetry of this ODE and, as a matter of fact, $i_{U_1}\alpha = 0$. Choosing a combination ω^1 of α^0 and α^1 such that $i_{U_1}\omega^1 = 1$ and taking $\omega^2 = W\alpha$, one, therefore, obtains a solvable structure $\{\omega^1, \omega^2\}$ for the second-order ODE given by $y^{(2)} = h(x, y, y^{(1)})$.

We summarize the results for the eight second-order ODEs constructed by Guo and Abraham-Schrauner in [2] in table 1. For legibility's sake, we substitute p for $y^{(1)}$. In the first column, we list $h(x, y, p)$ which defines the ODE by $y^{(2)} = h(x, y, p)$. In the second and third columns, we give ω^1 and ω^2 in terms of the one-forms $\alpha^0 = \mathrm{d}y - p\,\mathrm{d}x$ and $\alpha^1 = \mathrm{d}p - h\,\mathrm{d}x$ which represent the ODE. The last column contains $\mathrm{d}\omega^1$ in order to show that $\{\omega^1, \omega^2\}$ do form a solvable structure.

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